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New Determination of Zonal Harmonics Coefficients of the Earth's Gravitational Potential

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Abstract.—From Baker-Nunn observations of nine satellites, whose inclinations cover a region between 28° and 95°, the following values were derived for the zonal harmonics coefficients of the earth's gravitational field:

$J_2 = 1082.645\times10^{-6}$, ±6	$J_3 = -2.546\times10^{-6}$, ±20
$J_4 = -1.649\times10^{-6}$, ±16	$J_5 = -0.210\times10^{-6}$, ±25
$J_6 = 0.646\times10^{-6}$, ±30	$J_7 = -0.333\times10^{-6}$, ±39
$J_8 = -0.270\times10^{-6}$, ±50	$J_9 = -0.053\times10^{-6}$, ±60
$J_{10} = -0.054\times10^{-6}$, ±50	$J_{11} = 0.302\times10^{-6}$, ±35
$J_{12} = -0.357\times10^{-6}$, ±47	$J_{13} = -0.114\times10^{-6}$, ±84
$J_{14} = 0.179\times10^{-6}$, ±63	

1. INTRODUCTION

IN A PREVIOUS PAPER (Kozai, 1963) I derived a set of values for the coefficients of zonal spherical harmonics in the earth's grativational potential from the available observations of artificial satellites. However, at that time I did not give much weight to observations of high-inclination satellites simply because accurate observations for such satellites were not available.

We now have precisely reduced Baker-Nunn observations for some of the high-inclination satellites, and I have found that secular motions of ascending nodes of these satellites cannot be accurately expressed by my previous values of zonal harmonics. Therefore, I had to improve my previous values by adding observations of the high-inclination satellites and higher-order harmonics to the expression of the earth's potential.

In this paper I have tried to eliminate any accidental errors in observational data, by using many more observations of a given satellite than in my previous paper. I have used fourteen sets of observations for 1959 $\alpha 1$ and ten sets for 1959η, in contrast to the single set of data used for each satellite previously. Consequently, I believe that the data reported here are more reliable than those in the previous paper even for low-inclination satellites. Although we still lack sufficient observations for satellites with inclinations of between fifty and eighty degrees, this gap in the data will probably be filled in the near future.

2. METHOD OF REDUCTION

The observations used in this determination were made by Baker-Nunn cameras, and the first steps in the reductions were made by Phyllis Stern by the Differential Orbit Improvement program, in which first-order short-periodic perturbations due to the oblateness of the earth are taken out. The mean orbital elements of each satellite for every two days or four days were obtained from observations covering four or eight days. Luni-solar periodic and solar radiation perturbations in the orbital elements were then computed and subtracted from the mean orbital elements.

To derive secular motions of the ascending node and the perigee and amplitudes of long-periodic terms from these orbital elements, I use data covering about one period of revolution of argument of perigee, that is, about 80 days for Vanguard satellites, for example.

Secular accelerations in the mean anomaly or the mean longitude, and secular decreases in the semimajor axis due to air-drag, are then evaluated roughly; they can be used to compute theoretically secular variation in the longitude of the ascending node, the argument of perigee; and the eccentricity due to the air drag with sufficient accuracy, by assuming the rate of secular decrease of the perigee height. The computed secular variations in the three orbital elements are subtracted from the mean elements.

After the corrections with long-periodic perturbations due to even zonal harmonic terms are made, the argument of perigee ω, the longitude of the ascending node, Ω, the inclination i, and the eccentricity e are expressed by the following simple forms:

$$
\begin{aligned}
\omega &= \omega_0 + \dot{\omega} t + A_\omega \cos \omega, \\
\Omega &= \Omega_0 + \dot{\Omega} t + A_\Omega \cos \omega, \\
i &= i_0 + A_i \sin \omega, \\
e &= e_0 + A_e \sin \omega.
\end{aligned}
\tag{1}
$$

By the method of least squares we can determine the constants appearing in the formulas (1) from a set of the corrected orbital elements. However, when the eccentricity is very small, say less than 0.02, the corrected eccentricity and the argument of perigee are more accurately expressed by the following formulas:

$$
\left.
\begin{aligned}
e \sin \omega &= e_0 (1-\alpha) \sin (\omega_0 + \dot{\omega} t) + A_e, \\
e \cos \omega &= e_0 (1+\alpha) \cos (\omega_0 + \dot{\omega} t),
\end{aligned}
\right\}
\tag{2}
$$

where α, which is due to even-order harmonics, can be computed with approximate values of J_n as

$$
\begin{aligned}
\alpha = \sin^2 i \{ & J_2^2 (14-15 \sin^2 i) + 5 J_4 (6-7 \sin^2 i) \\
& - 10.9375 J_6 (16-48 \sin^2 i + 33 \sin^4 i)/a^2 \} / \{ 16 a^2 J_2 (4-5 \sin^2 i) \}.
\end{aligned}
\tag{3}
$$

By using the formulas (2) we can determine $e_0 \sin \omega_0$, $e_0 \cos \omega_0$, A_e and a correction to an assumed value of $\dot{\omega}$ from observations by the method of least squares.

The relation between the anomalistic mean motion n and our semimajor axis a is given as

$$
n^2 a^3 = GM \left\{ 1 + \frac{3 J_2}{4 p^2} (1-e^2)^{\frac{1}{2}} (1-3 \cos^2 i) \right\},
\tag{4}
$$

where

$$
\begin{aligned}
GM &= 3.986032 \times 10^{20} \text{ cm}^3/\text{sec}^2, \\
p &= a(1-e^2).
\end{aligned}
\tag{5}
$$

Expressing the mean motion in revolutions per day and the semimajor axis in earth's equatorial radii, we can use the following number for GM:

$$
\sqrt{GM} = 17.043570,
\tag{6}
$$

where I adopt the following value of the equatorial radius:

$$
a_e = 6378.165 \text{ km}.
\tag{7}
$$

Table 1.—*Orbital Data for 1959 Alpha 1*

	Epoch	n	i_0	e_0	$\dot{\omega}$	$\dot{\Omega}$	A_e	A_i	A_ω	A_Ω
1	Apr. 2, 1959	4120°.861	32°.879 60	0.165 654	5°.262 32	−3°.500 307	0.469×10⁻³	−0°.677×10⁻²	0°.1600	0°.69×10⁻²
			±10	±4	±17	±18	±7	±17	±40	±5
2	June 21, 1959	4123 .878	.879 71	358	5 .270 05	−3 .505 504	0.474	−0 .715	0 .1516	0. 96
			±14	±3	±8	±21	±5	±18	±23	±7
3	Sept. 17, 1959	4125 .316	.880 02	283	5 .274 05	−3 .508 239	0.475	−0 .701	0 .1548	0 .40
			±16	±3	±8	±21	±6	±33	±36	±9
4	Dec. 6, 1959	4125 .995	.879 94	162	5 .275 48	−3 .509 301	0.459	−0 .647	0 .1618	0 .54
			±12	±5	±11	±16	±6	±16	±29	±4
5	Mar. 7, 1960	4126 .610	.879 28	0.164 958	5 .276 87	−3 .510 082	0.464	−0 .690	0 .1559	0 .76
			±5	±3	±6	±11	±4	±7	±17	±3
6	May 24, 1960	4127 .508	.878 44	763	5 .278 93	−3 .511 425	0.457	−0 .717	0 .1533	0 .87
			±10	±2	±3	±7	±4	±14	±12	±3
7	Aug. 22, 1960	4128 .877	.878 98	642	5 .282 59	−3 .513 799	0.464	−0 .642	0 .1534	0 .88
			±10	±2	±2	±13	±5	±13	±10	±6
8	Nov. 26, 1960	4130 .345	.879 47	577	5 .286 55	−3 .516 572	0.464	−0 .662	0 .1621	0 .57
			±15	±2	±6	±15	±3	±21	±18	±5
9	Feb. 18, 1961	4130 .675	.879 32	567	5 .287 59	−3 .517 208	0.453	−0 .680	0 .1560	0 .63
			±18	±3	±4	±12	±5	±30	±13	±4
10	May 13, 1961	4130 .769	.879 25	447	5 .287 40	−3 .517 094	0.460	−0 .667	0 .1564	0 .78
			±9	±2	±2	±12	±2	±11	±8	±5
11	Aug. 13, 1961	4130 .948	.878 48	330	5 .287 68	−3 .517 199	0.455	−0 .705	0 .1557	0 .85
			±5	±1	±2	±7	±2	±7	±10	±2
12	Nov. 17, 1961	4131 .525	.878 98	292	5 .289 317	−3 .518 265	0.465	−0 .702	0 .1577	0 .75
			±4	±1	±13	±6	±3	±7	±8	±4
13	Feb. 13, 1962	4131 .834	.878 94	378	5 .290 46	−3 .519 099	0.462	−0 .698	0 .1565	0 .41
			±11	±2	±2	±2	±3	±14	±14	±4
14	June 3, 1962	4132 .021	.879 44	396	5 .291 059	−3 .519 495	0.463	−0 .647	0 .1600	0 .56
			±8	±1	±19	±6	±2	±12	±9	±3

The earth's gravitational potential is expressed with Legendre polynomials as

$$U=\frac{GM}{r}\left\{1-\sum_{n=2}^{\infty} J_n (a_e/r)^n P_n(\sin\beta)\right\} \tag{8}$$

The secular motions of the node and the perigee and the amplitudes of long-periodic terms with argument ω derived from observations are compared with those computed from my previous value of J_n (Kozai, 1963),

$$\begin{aligned} J_2&=1082.48\times10^{-6}, & J_3&=-2.562\times10^{-6},\\ J_4&=-1.84\times10^{-6}, & J_5&=-0.064\times10^{-6},\\ J_6&=0.39\times10^{-6}, & J_7&=-0.470\times10^{-6},\\ J_8&=-0.02\times10^{-6}, & J_9&=0.117\times10^{-6}. \end{aligned} \tag{9}$$

Of course we must include luni-solar secular terms and a J_2^2 term, which can be computed with an approximate value of J_2 to compute secular motions. Therefore, each secular motion and amplitude provides us with (O–C), which will make it possible to improve values of J_n.

3. DATA

(a) 1959 Alpha 1—Table 1 lists fourteen sets of data for this satellite, and table 2 gives (O–C)'s referred to my previous values for J_n.

The standard deviations for the daily secular motions $\dot{\omega}$ and $\dot{\Omega}$ given in table 1 are determined from observations; those in table 2 are computed by adding uncertainties which come from those in e_0 and i_0. Weighted mean values for the fourteen sets are given at the bottom of the table. As can be seen, the scattering of (O–C)'s is much larger than that expected from the standard deviations assigned to the observed values. However, the standard deviations assigned to the mean values in table 2 should be more reliable, and will be used in the determinations of J_n.

(b) 1959 Eta—Ten sets of data are given in tables 3 and 4 for this Vanguard satellite. However, its orbital elements are not essentially different from those of 1959 α1 and the mean values of (O–C) in table 4 are almost identical with those in table 2, as expected. For the two Vanguard satellites (O–C) in $\dot{\Omega}$ and A_ω are significantly large.

(c) 1960 Iota 2—Since the eccentricity is very small for this rocket of Echo I, the formulas (2) are used in the reduction. Since $\dot{\omega}+\dot{\Omega}$ are very small for this satellite, it is necessary to take special care to compute terms with arguments $2(\omega+\Omega-\Omega_\odot)$ and $2(\omega+\Omega-\Omega_{☾})$ in the luni-solar perturbations.

Five sets of data are given in tables 5 and 6. For this satellite the scattering of (O–C) for secular motions is very large. The large scatter-

Table 2.—(O–C) *Referred to Kozai's Previous Constants for 1959 Alpha 1*

	$\dot{\omega}\times10^6$	$\dot{\Omega}\times10^6$	$A_e\times10^6$	$A_i\times10^5$	$A_\omega\times10^4$	$A_\Omega\times10^4$
1	19° ± 17°	−31° ± 18°	12 ± 7	13° ± 17°	90° ± 40°	−9° ± 5°
2	0 ± 8	49 ± 23	17 ± 5	−26 ± 18	2 ± 23	19 ± 7
3	3 ± 8	−23 ± 23	18 ± 6	−13 ± 33	33 ± 36	−37 ± 9
4	−15 ± 11	−22 ± 21	2 ± 6	41 ± 16	102 ± 29	−23 ± 4
5	3 ± 6	−42 ± 13	7 ± 4	−3 ± 7	41 ± 17	−1 ± 3
6	−3 ± 4	−33 ± 10	0 ± 4	−31 ± 14	12 ± 12	10 ± 3
7	6 ± 3	3 ± 14	7 ± 5	44 ± 13	12 ± 10	11 ± 6
8	−5 ± 7	−27 ± 17	7 ± 3	24 ± 21	98 ± 18	−20 ± 5
9	1 ± 5	−24 ± 16	−4 ± 5	6 ± 30	37 ± 13	−14 ± 4
10	−4 ± 3	−8 ± 15	3 ± 2	18 ± 11	40 ± 8	1 ± 5
11	0 ± 3	−4 ± 8	−2 ± 2	−20 ± 7	31 ± 10	8 ± 2
12	13 ± 3	−33 ± 7	8 ± 3	−17 ± 7	51 ± 8	−2 ± 4
13	3 ± 3	−46 ± 14	5 ± 3	−13 ± 14	40 ± 14	−36 ± 4
14	9 ± 2	−48 ± 8	6 ± 2	38 ± 12	75 ± 9	−21 ± 3
Mean	4 ± 2	−26 ± 6	4 ± 2	−2 ± 8	42 ± 8	−5 ± 5

ing for $\dot{\omega}$ may be partly due to the fact that the radiation pressure effects in the argument of perigee are too large to handle accurately. Also, I suspect that the anomalistic mean motion cannot be determined with sufficient accuracy for a satellite of such small eccentricity. This might be one reason why we have large discrepancies in the secular motions of the node.

However, (O–C)'s in $\dot{\omega}$, $\dot{\Omega}$ and A_e are still significant.

(d) 1961 Nu—For this satellite precisely reduced Baker-Nunn observations are not available and observations must be used that are not precisely reduced. However, since the satellite is close to the earth and the inclination is the smallest used in this paper, the node and the perigee move rapidly and the relative accuracies in the determination of the secular motions are fair.

Four sets of data are given in tables 7 and 8, which show a wide scatter in the values of (O–C) in A_e and A_i. The residuals in the two secular motions take large values. This satellite was not used in the earlier determination of J_n; at that time the smallest inclination was 32°.9, for 1959 α1.

(e) 1961 Omicron—There are two separate satellites for 1961 o. However, since they have almost identical orbital elements, they are treated as one satellite here. The eccentricity is very small. Since the inclination is rather close to the critical inclination, the argument of perigee moves very slowly. Therefore, one set of observations must cover more than 500 days. However, as the mean motion changes rather rapidly due to air drag, I have used one set of 400-day observations.

Table 3.—*Orbital Data for 1959 Eta*

	Epoch	n	i_0	e_0	$\dot{\omega}$	$\dot{\Omega}$	A_e	A_i	A_ω	A_Ω
1	Nov. 4, 1959	3982°.496	33°.355 10	0.190 019	4°.872 23	−3°.272 67	0.442×10^{-3}	$-0°.820\times10^{-2}$	0°.1330	$0°.90\times10^{-2}$
			±15	±2	±7	±3	±5	±30	±25	±8
2	Feb. 4, 1960	3983 .406	.354 70	0.189 782	4 .874 03	−3 .273 818	0.451	−0 .760	0 .1299	0 .81
			±12	±3	±9	±9	±4	±20	±10	±2
3	May 4, 1960	3984 .637	.354 01	519	4 .876 67	−3 .275 477	0.441	−0 .759	0 .1327	0 .90
			±11	±2	±3	±6	±4	±16	±12	±2
4	Aug. 2, 1960	3986 .079	.353 34	326	4 .880 08	−3 .277 815	0.450	−0 .793	0 .1323	0 .93
			±10	±2	±3	±9	±3	±15	±11	±4
5	Nov. 10, 1960	3988 .708	.353 82	075	4 .886 54	−3 .282 159	0.451	−0 .817	0 .1358	0 .82
			±12	±3	±3	±10	±3	±18	±18	±6
6	Feb. 22, 1961	3989 .670	.354 72	059	4 .889 27	−3 .284 07	0.462	−0 .700	0 .1320	0 .86
			±20	±2	±4	±3	±5	±40	±15	±8
7	June 18, 1961	3989 .952	.354 36	001	4 .889 75	−3 .284 384	0.461	−0 .687	0 .1319	0 .83
			±13	±2	±2	±12	±2	±19	±11	±6
8	Oct. 16, 1961	3990 .168	.353 44	0.188 824	4 .889 70	−3 .284 302	0.455	−0 .770	0 .1335	0 .75
			±7	±2	±2	±11	±4	±9	±7	±3
9	Jan. 14, 1962	3990 .437	.354 42	742	4 .890 35	−3 .284 641	0.447	−0 .839	0 .1350	0 .87
			±8	±2	±2	±23	±2	±11	±11	±2
10	Apr. 22, 1962	3991 .063	.353 25	696	4 .891 779	−3 .285 739	0.467	−0 .806	0 .1353	0 .76
			±8	±2	±11	±3	±3	±13	±4	±2

Table 4.—(O–C) *for 1959 Eta*

	$\dot{\omega}\times10^5$	$\dot{\Omega}\times10^6$	$A_e\times10^6$	$A_i\times10^5$	$A_\omega\times10^4$	$A_\Omega\times10^4$
1	9° ± 8°	−70° ±31°	−9 ±5	−50° ±30°	50° ±25°	0° ±8°
2	14 ± 4	−70 ±13	0 ±4	10 ±20	17 ±10	−9 ±2
3	17 ± 4	−20 ± 9	−9 ±4	14 ±16	43 ±12	0 ±2
4	9 ± 4	−62 ±11	−2 ±3	−21 ±15	37 ±11	3 ±4
5	8 ± 4	−23 ±14	−1 ±3	−46 ±14	70 ±18	−8 ±6
6	26 ± 5	−160 ±40	10 ±5	7 ± 4	32 ±15	−4 ±8
7	10 ± 3	−70 ±14	9 ±2	84 ±19	30 ±11	−7 ±6
8	−4 ± 3	7 ±13	3 ±4	0 ± 9	45 ± 7	−15 ±3
9	32 ±20	−66 ±24	−5 ±2	−69 ±11	59 ±11	−3 ±2
10	−5 ± 3	−34 ± 7	15 ±3	−37 ±13	62 ± 4	−14 ±2
Mean	7 ± 3	−40 ± 9	2 ±2	−4 ± 9	50 ± 5	−7 ±2

Table 5.—*Orbital Data for 1960 Iota 2*

	Epoch	n	i_0	e_0	$\dot{\omega}$	$\dot{\Omega}$	A_ω	A_i	A_Ω
1	Nov. 12, 1960	4390°.918	47°.231 76	0.011 475	2°.977 64	−3°.101 208	0.6572×10⁻³ → 0.6572×10^{-3}	$-0°.32\times10^{-3}$	$0°.18\times10^{-2}$
			±7	±1	±17	±3	±14	±11	±3
2	Mar. 12, 1961	4390°.915	.231 63	265	2 .978 32	−3 .101 186	0.6616	−0 .46	0 .03
			±8	±1	±21	±4	±14	±11	±3
3	July 10, 1961	4390 .893	.231 31	412	2 .977 76	−3 .101 200	0.6651	−0 .48	0 .00
			±7	±1	±18	±3	±14	±10	±3
4	Nov. 7, 1961	4390 .898	.231 92	490	2 .977 13	−3 .101 239	0.6600	−0 .40	0 .19
			±6	±1	±24	±3	±20	±10	±3
5	Mar. 7, 1962	4390 .923	.232 49	373	2 .978 82	−3 .101 192	0.6629	−0 .77	0 .13
			±5	±1	±14	±2	±11	±7	±2

Table 6.—*(O–C) for 1960 Iota 2*

	$\dot{\omega}\times 10^5$	$\dot{\Omega}\times 10^6$	$A_e\times 10^7$	$A_i\times 10^5$	$A_\Omega\times 10^4$
1	67° ±17°	−31° ± 5°	44 ±14	8° ±11°	8° ±3°
2	134 ±21	−16 ± 6	88 ±14	−7 ±11	−2 ±3
3	75 ±18	−46 ± 5	123 ±14	−9 ±10	−10 ±3
4	22 ±24	−101 ± 5	72 ±20	0 ±10	10 ±3
5	200 ±24	−64 ± 4	101 ±11	−38 ± 7	4 ±3
Mean	90 ±30	−52 ±15	86 ±13	−9 ± 8	2 ±4

Table 7.—*Orbital Data for 1960 Nu*

	Epoch	n	i_0	e_0	$\dot{\omega}$	$\dot{\Omega}$	A_e	A_i	A_ω	A_Ω
1	May 12, 1961	4798°.022	28°.803 9	0.086 211	8°.102 98	−5°.003 817	0.452×10^{-3}	$-5°.98\times 10^{-3}$	0°.3054	$0°.50\times 10^{-3}$
			±3	±2	±13	±27	±3	±44	±25	±5
2	June 18, 1961	4798 .224	.803 9	197	8 .103 59	−5 .004 400	0.455	−5 .60	0 .3050	0 .49
			±2	±2	±10	±20	±3	±30	±40	±6
3	July 12, 1961	4798 .355	.804 1	195	8 .104 28	−5 .004 728	0.452	−5 .30	0 .3075	0 .54
			±3	±3	±15	±75	±5	±35	±34	±11
4	Sept. 6, 1961	4798 .561	.804 6	159	8 .104 20	−5 .005 105	0.480	−4 .43	0 .3166	0 .32
			±2	±7	±22	±41	±8	±24	±61	±12

For this satellite, the mean height is rather low, about 900 km, and the inclination is high. Therefore, the object is rather difficult to observe from the Baker-Nunn stations due to visibility conditions, and there are many gaps in the observations, periods for which accurate orbital elements are not available. As the Baker-Nunn stations are between $+35°$ and $-35°$ in latitude, the inclination of this satellite is poorly determined although the longitude of the node can be well determined. This situation is contrary to that of Vanguard satellites.

Table 8.—*(O–C) for 1960 Nu*

	$\dot{\omega}\times 10^5$	$\Omega\times 10^6$	$A_e\times 10^6$	$A_i\times 10^5$	$A_\omega\times 10^4$	$A_\Omega\times 10^4$
1	$-59°\pm 15°$	$211°\pm 30°$	-17 ± 3	$-173°\pm 44°$	$-45°\pm 20°$	$17°\pm 5°$
2	-74 ± 10	90 ± 20	-14 ± 3	-140 ± 30	-50 ± 40	16 ± 6
3	-51 ± 15	71 ± 80	-17 ± 3	-105 ± 35	-25 ± 34	21 ± 11
4	-10 ± 22	111 ± 45	11 ± 8	-20 ± 24	64 ± 61	0 ± 12
Mean	-48 ± 20	131 ± 40	-9 ± 14	-110 ± 70	-14 ± 50	14 ± 10

The secular motion of the node is determined quite accurately, as we can see in table 9. However, we cannot compute theoretical values of the secular motions so accurately as the observed ones, because of uncertainties in the inclination. Therefore, the standard deviations in (O–C) of $\dot{\Omega}$ in table 10 are large. But (O–C)'s in $\dot{\Omega}$ themselves are quite large, as we can see in table 10. In the previous determination of J_n, accurate orbital elements from Baker-Nunn observations were not available.

The value of (O–C) in $\dot{\Omega}$ for the epoch 4 is quite different from the others, and I suspect this scattering is due to some accidental errors in i_0 for the epoch 4, and give small weight to this value in taking the mean.

For this satellite the radiation pressure effect in the argument of perigee is too large for my program to compute it with enough accuracy. This is also true for other satellites of small eccentricity.

(f) 1961 Alpha Delta 1—This satellite has a polar orbit. However, as the mean height is quite high, we can determine the orbit very accurately from Baker-Nunn observations.

This satellite, and the three listed in tables 13–17, which were launched in 1962, were not used in my previous determination.

The first set of data is determined from 300-day observations, and the second set is from 400-day observations, which cover one revolution of argument of perigee.

To compute the solar-perturbations there arise three small divisors, namely, $2(n_\odot-\dot{\omega})$, $2(\dot{\omega}-\dot{\Omega}+n_\odot)$, and $2(n_\odot-2\dot{\omega}-\dot{\Omega})$.

Tables 11 and 12 show that the eccentricity is very small and that (C–C) in $\dot{\Omega}$ is very significant.

Table 9.—*Orbital Data for 1960 Omicron*

	Epoch	n	i_0	e_0	$\dot{\omega}$	$\dot{\Omega}$	A_e	A_i	A_Ω
	Omicron 1:								
1	Mar. 5, 1962	4993°.199	66°.815 73	0.008 022	−0°.695 76	−2°.424 778	0.250×10^{-3}	$-0°.04\times10^{-3}$	$0°.72\times10^{-2}$
			±12	±2	±23	±1	±3	±19	±3
2	Nov. 16, 1962	4993 .276	.815 28	0.007 981	−0 .695 20	−2 .424 864	0.266	−0 .77	0 .60
			±13	±2	±8	±1	±2	±16	±3
	Omicron 2:								
3	Mar. 1, 1962	4992 .762	.815 40	0.008 055	−0 .695 62	−2 .424 295	0.256	−0 .22	0 .75
			±10	±2	±18	±1	±3	±19	±4
4	Nov. 16, 1962	4992 .817	.815 50	026	−0 .695 63	−2 .424 349	0.264	0 .02	0 .66
			±15	±2	±11	±1	±2	±21	±2

Table 10.—*(O–C) for 1960 Omicron*

	$\dot{\omega}\times10^{-5}$	$\dot{\Omega}\times10^{-6}$	$A_e\times10^{6}$	$A_i\times10^{5}$	$A_\Omega\times10^{4}$
1	50° ±23°	−1291° ±12°	−51 ±3	2° ±19°	7° ±3°
2	54 ± 9	−1238 ±13	−35 ±2	−18 ±16	−5 ±2
3	−2 ± 8	−1257 ±10	−46 ±3	−16 ±19	10 ±4
4	−20 ±11	−1162 ±15	−37 ±2	8 ±21	2 ±2
Mean	20 ±30	−1262 ±25	−42 ±6	−6 ±13	4 ±8

Table 11.—*Orbital Data for 1961 Alpha Delta 1*

	Epoch	n	i_0	e_0	$\dot{\omega}$	$\dot{\Omega}$	A_e	A_i	A_Ω
1	Aug. 4, 1962	3123°.598	95°.856 47 ±5	0.012 092 ±1	−0°.976 93 ±11	0°.210 391 ±1	0.787×10⁻³ ±2	0°.51×10⁻³ ±7	−1°.25×10⁻³ ±25
2	Sept. 21, 1962	3123 .598	.856 69 ±6	0.012 073 ±2	−0 .977 97 ±10	0 .210 393 ±1	0.804×10⁻³ ±3	0 .24 ±8	0 .54×10⁻³ ±20

Table 12.—*(O–C) for 1961 Alpha Delta 1*

	$\dot{\omega}\times10^5$	$\dot{\Omega}\times10^6$	$A_e\times10^6$	$A_i\times10^5$	$A_\Omega\times10^4$
1	33° ±11°	68° ±2°	4 ±2	−5 ± 7	−7 ±2
2	−72 ±10	63 ±2	21 ±3	18 ± 8	7 ±3
Mean	−20 ±50	65 ±2	8 ±4	7 ±10	7 ±7

Table 13.—*Orbital Data for 1962 Alpha Epsilon*

	Epoch	n	i_0	e_0	$\dot{\omega}$	$\dot{\Omega}$	A_e	A_i	A_ω	A_Ω
1	Oct. 7, 1962	3285°.400	44°.799 53 ±6	0.242 241 ±1	1°.986 171 ±8	−1°.858 849 ±4	0.5461×10⁻³ ±16	−0°.761×10⁻² ±9	0°.1117 ±5	0°.0176 ±3
2	Oct. 15, 1962	3285 .401	.799 13 ±15	239 ±3	1 .986 179 ±12	−1 .858 849 ±4	0.5506 ±35	−0 .721 ±21	0 .1116 ±9	0 .0180 ±3
3	Apr. 17, 1963	3285 .424	.800 85 ±9	319 ±3	1 .986 074 ±7	−1 .858 983 ±3	0.5697 ±39	−0 .697 ±11	0 .1124 ±9	0 .0179 ±3

Table 14.—*(O–C) for 1962 Alpha Epsilon*

	$\dot{\omega}\times10^5$	$\dot{\Omega}\times10^6$	$A_e\times10^6$	$A_i\times10^5$	$A_\omega\times10^4$	$A_\Omega\times10^4$
1	47°.3 ± 1°.0	−60° ± 5°	38 ± 2	8° ± 9°	−22° ± 5°	31° ± 3°
2	44 .0 ± 2 .2	−51 ± 7	32 ± 4	49 ± 21	−24 ± 9	35 ± 3
3	33 .4 ± 1 .0	−57 ± 8	52 ± 4	73 ± 11	−15 ± 9	35 ± 3
Mean	42 ± 6	−56 ± 5	37 ± 20	43 ± 33	−20 ± 5	34 ± 3

Table 15.—*Orbital Data for 1962 Beta Mu 1*

	Epoch	n	i_0	e_0	$\dot{\omega}$	Ω	A_e	A_i	A_Ω
1	Jan. 5, 1963	4804°.149	50°.141 05	0.007 060	2°.964 39	−3°.609 041	0.7822×10^{-3}	$-0°.73\times10^{-3}$	$0°.99\times10^{-3}$
			±15	±2	±61	±10	±20	±25	±34
2	May 5, 1963	4804 .152	.142 46	055	2 .960 06	−3 .609 023	0.7745	−0 .53	1 .95
			±18	±2	±55	±8	±22	±23	±48
3	Mar. 6, 1963	4804 .150	.141 79	060	2 .963 41	−3 .608 983	0.7757	−1 .18	1 .62
			±13	±2	±33	±5	±17	±17	±37

Table 16.—(O–C) *for 1962 Beta Mu 1*

	$\dot{\omega}\times10^5$	$\dot{\Omega}\times10^6$	$A_e\times10^6$	$A_i\times10^5$	$A_\Omega\times10^4$
1	271° ± 61°	16° ± 14°	27 ± 2	−47° ± 25°	1° ± 3°
2	−128 ± 55	−69 ± 18	19 ± 2	−27 ± 23	11 ± 5
3	191 ± 33	18 ± 15	20 ± 2	−92 ± 17	8 ± 4
Mean	131 ± 150	−12 ± 30	22 ± 4	−55 ± 30	7 ± 5

Table 17.—*Orbital Data and (O–C) for 1962 Beta Upsilon*

Epoch	n	i_0	e_0	$\dot{\omega}$	$\dot{\Omega}$	A_e	A_i	A_ω	A_Ω
Apr. 1, 1963	2801°.146	47°.510 10 ±18	0.284 224 ±2	1°.212 096 ±12	−1°.279 119 ±4	0.521×10^{-3} ±4	$-0°.867\times10^{-2}$ ±27	0°.0966 ±20	0°.0246 ±6

Epoch	$\dot{\omega}\times10^5$	$\dot{\Omega}\times10^6$	$A_e\times10^6$	$A_i\times10^5$	$A_\omega\times10^4$	$A_\Omega\times10^4$
Apr. 1, 1963	30° ± 7°	−156° ± 7°	18 ± 4	−51° ± 27°	4° ± 20°	88° ± 6°

(g) 1962 Alpha Epsilon—For this satellite three sets of data are given in table 13. However, observations in sets 1 and 2 are overlapped widely. Since ω and $-\Omega$ have nearly the same value, $2(\omega+\Omega)$ and $2(n_\odot-2\omega-\Omega)$ take small values, as for 1962 Beta Mu 1. Therefore we must be careful to compute luni-solar perturbation terms with such arguments.

All values of (O–C) in table 14 are significant.

(h) 1962 Beta Mu 1—This is a geodetic satellite, and although the inclination is not very much different from that of 1956 $\alpha\epsilon$, the eccentricity and the mean motion take quite different values.

The mean height of this satellite is not high enough for the Baker-Nunn cameras to track the object over a long arc. Therefore the accuracy of determination of the orbital elements is not high.

(i) 1962 Beta Upsilon—Unfortunately, precisely reduced Baker-Nunn observations are available for this satellite only for 200 days, during which the argument of perigee moves by 240°. Therefore I will increase by a factor of five the standard deviations given in table 17 in the determination of J_n.

4. DETERMINATION OF J_n

Table 18 gives for the nine satellites the semimajor axes in units of earth equatorial radii, the inclinations, the eccentricity, and the area-to-mass ratio in cgs units. The same table also gives J_2^2 terms and luni-solar secular terms in $\dot\omega$ and $\dot\Omega$ (Kozai, 1962; Kozai, 1959).

A previous paper (Kozai, 1962) gives the formulas used to compute secular perturbations and amplitudes of long-periodic terms with argument ω by including up to 8th-order harmonics. However, I include up to 14th-order harmonics in the present determination, and the additional formulas are given in the following:

$$
\begin{aligned}
\delta\dot\Omega = &-\frac{3465 J_{10}}{4{,}194{,}304 p^{10}}\,\theta n(63-1092\theta^2+4914\theta^4-7956\theta^6+4199\theta^8)\\
&\times(128+2304e^2+6048e^4+3360e^6+315e^8)\\
&-\frac{9009 J_{12}}{67{,}108{,}864 p^{12}}\,\theta n(231-5775\theta^2+39270\theta^4-106{,}590\theta^6\\
&+124{,}355\theta^8-52{,}003\theta^{10})\cdot(256+7040e^2+31{,}680e^4+36{,}960e^6\\
&+11{,}550e^8+693e^{10})\\
&-\frac{45{,}045 J_{14}}{2{,}147{,}483{,}648 p^{14}}\,n\theta(429-14{,}586\theta^2+138{,}567\theta^4-554{,}268\theta^6\\
&+1{,}062{,}347\theta^8-965{,}770\theta^{10}+334{,}305\theta^{12})\cdot(1024+39{,}936e^2\\
&+274{,}560e^4+549{,}120e^6+360{,}360e^8+72{,}072e^{10}+3003e^{12}),
\end{aligned}
\tag{10}
$$

Table 18.—*Summary of Parameters*

	a	i	e	J_2^2 in $\dot{\omega}$	$\odot + ☾$ in $\dot{\omega}$	J_2^2 in $\dot{\Omega}$	$\odot + ☾$ in $\dot{\Omega}$	A/M
(a) 59 α1	1.30	32°.8	0.16	2°.04×10⁻³	0°.55×10⁻³	0°.305×10⁻³	−0°.378×10⁻³	0.21
(b) 59 η1	1.33	33 .4	0.19	1 .91	0 .57	0 .223	−0 .395	0.27
(c) 60 ι2	1.25	47 .2	0.01	2 .02	0 .26	−0 .705	−0 .272	0.21
(d) 61 ν	1.18	28 .8	0.09	2 .77	0 .52	1 .092	−0 .327	0.15
(e) 61 o	1.15	66 .8	0.01	−0 .87	−0 .04	−1 .875	−0 .139	0.08
(f) 61 $\alpha\delta$1	1.57	95 .9	0.01	−0 .77	−0 .27	0 .114	0 .058	0.06
(g) 62 $\alpha\epsilon$	1.52	44 .8	0.24	1 .01	0 .44	−0 .286	−0 .427	0.08
(h) 62 $\beta\mu$1	1.18	50 .1	0.01	2 .48	0 .19	−1 .209	−0 .235	0.07
(i) 62 $\beta\upsilon$	1.69	47 .5	0.28	0 .58	0 .45	−0 .227	−0 .497	0.08

$$\delta\dot{\omega} = -\theta\delta\dot{\Omega}$$

$$-\frac{3465J_{10}}{8{,}388{,}608p^{10}}\, n(63-3465\theta^2+30{,}030\theta^4-90{,}090\theta^6+109{,}395\theta^8$$

$$-46{,}189\theta^{10})\cdot(128+1152e^2+2016e^4+840e^6+63e^8)$$

$$-\frac{9009J_{12}}{268{,}435{,}456p^{12}}\, n(231-18{,}018\theta^2+225{,}225\theta^4-1{,}021{,}020\theta^6$$

$$+2{,}078{,}505\theta^8-1{,}939{,}938\theta^{10}+676{,}039\theta^{12})\cdot(1024+39{,}936e^2$$

$$+274{,}560e^4+549{,}120e^6+360{,}360e^8+72{,}072e^{10}+3003e^{12}) \tag{11}$$

$$-\frac{45{,}045J_{14}}{4{,}294{,}967{,}296p^{14}}\, n(429-45{,}045\theta^2+765{,}765\theta^4-4{,}849{,}845\theta^6$$

$$+14{,}549{,}535\theta^8-22{,}309{,}287\theta^{10}+16{,}900{,}975\theta^{12}-5{,}014{,}575\theta^{14})$$

$$\times(1024+19{,}968e^2+91{,}520e^4+137{,}280e^6+72{,}072e^8$$

$$+12{,}012e^{10}+429e^{12}),$$

$$\delta e = -\sin i\ (1-5\theta^2)^{-1}(1-e^2)\sum_{j=4}^{6} C_jA_jB_j \sin\omega, \tag{12}$$

$$\delta i = -e\theta\ \delta e/\{\sin i\ (1-e^2)\}, \tag{13}$$

$$\delta\Omega = e\theta \sin^{-1} i\ (1-5\theta^2)^{-1}\sum_{j=4}^{6} C_j\{-\sin^2 i\cdot D_j$$

$$+(9-5\theta^2)(1-5\theta^2)^{-1}A_j\}B_j\cos\omega, \tag{14}$$

$$\delta\omega = -\theta\delta\Omega - \sin i\cdot e^{-1}\cdot(1-5\theta^2)^{-1}\sum_{j=4}^{6} C_jA_jE_j\cos\omega, \tag{15}$$

where

$\theta = \cos i$

$$C_4 = \frac{105J_9}{65{,}536J_2p^7}$$

$$C_5 = \frac{1155J_{11}}{4{,}194{,}304J_2p^9}$$

$$C_6 = \frac{3003J_{13}}{67{,}108{,}864J_2p^{11}}$$

$$A_4 = 7-308\theta^2+2002\theta^4-4004\theta^6+2431\theta^8$$

$$A_5 = 21-1365\theta^2+13{,}650\theta^4-46{,}410\theta^6+62{,}985\theta^8-29{,}393\theta^{10}$$

$$A_6 = 33-2970\theta^2+42{,}075\theta^4-213{,}180\theta^6+479{,}655\theta^8-490{,}314\theta^{10}+185{,}725\theta^{12}$$

$$B_4 = 64 + 336e^2 + 280e^4 + 35e^6$$

$$B_5 = 128 + 1152e^2 + 2016e^4 + 840e^6 + 63e^8$$

$$B_6 = 512 + 7040e^2 + 21{,}120e^4 + 18{,}480e^6 + 4620e^8 + 231e^{10}$$

$$D_4 = 88(7 - 91\theta^2 + 273\theta^4 - 221\theta^6)$$

$$D_5 = 130(21 - 420\theta^2 + 2142\theta^4 - 3876\theta^6 + 2261\theta^8)$$

$$D_6 = 60(99 - 2805\theta^2 + 21{,}318\theta^4 - 63{,}954\theta^6 + 81{,}719\theta^8 - 37{,}145\theta^{10})$$

$$E_4 = 64 + 1776e^2 + 4760e^4 + 2485e^6 + 210e^8$$

$$E_5 = 128 + 5504e^2 + 26{,}208e^4 + 30{,}072e^6 + 8{,}967e^8 + 504e^{10}$$

$$E_6 = 512 + 31{,}360e^2 + 232{,}320e^4 + 467{,}280e^6 + 300{,}300e^8 + 57{,}982e^{10} + 2310e^{12} \tag{16}$$

(a) Even harmonics—Table 18 gives equations of condition to determine values of J_2 through J_{14}. There are 18 equations with 7 unknowns. The equations can be solved by assigning to each a weight reciprocally proportional to the standard deviation. Actually, each equation is divided by its standard deviation, and then normal equations are constructed. Before solving the equations, note that $\Sigma(\text{O–C})^2$ is 3882 $(=18\times 14.7^2)$; that is (O–C) is bigger than the standard deviation by factor of 14.7. This value comes down to $23 = (18-6)\times 1.4^2$ after solving J_{12}, and to $13.4 = (18-7)\times 1.1^2$ after solving J_{14}, whereas it is $93.5 = (18-5)\times 2.7^2$ after J_{10} is solved. Therefore we can stop either at J_{12} or at J_{14}, although the solution including J_{14} is, of course, better.

In table 19 residuals based on the solutions up to J_{14} and J_{12} are given under headings I and II, respectively, in units of 10^{-6} degrees. Under the heading KH, residuals based on King-Hele and Cook's values (1964) are given; that is,

$$\begin{aligned} J_2 &= 1802.70\times 10^{-6}, & J_4 &= -1.40\times 10^{-6}, \\ J_6 &= 0.37\times 10^{-6}, & J_8 &= 0.07\times 10^{-6}, \\ J_{10} &= -0.50\times 10^{-6}, & J_{12} &= 0.31\times 10^{-6}. \end{aligned} \tag{17}$$

In the node equations the residuals based on my new determinations for 1962 $\beta\upsilon$ are larger than the standard deviations. However, since this datum is not entirely reliable, being based on a single determination covering an incomplete period of time, this may not be a weak point in this determination.

In the perigee equations of 1961 ν and 1962 $\alpha\epsilon$, the residuals are larger than their standard deviations. This may suggest that we must still include higher-order terms to express these data.

Table 19.—*Equations of Condition for Even Harmonics*

	J_2	J_4	J_6	J_8	J_{10}	J_{12}	J_{14}	(O–C)$\times 10^6$	I	II	KH
Perigee:											
(a)	4875	−1563	−2718	2481	410	−1916	934	40° ± 20°	5°	−2°	160°
(b)	4508	−1271	−2546	2124	530	−1744	711	70 ± 30	18	12	190
(c)	2753	2686	−1224	−2302	316	1425	37	910 ± 300	230	370	540
(d)	7476	−5168	−2589	6275	−3121	−2232	4333	−480 ± 200	−290	−360	−1610
(e)	−640	1895	4419	4324	1624	−1623	−1623	−200 ± 400	10	−660	340
(f)	−903	−637	−331	−144	−53	−15	−2	−200 ± 500	100	100	260
(g)	1835	1038	−821	−643	398	340	−202	420 ± 60	160	180	−310
(h)	2740	4130	−333	−4065	−1360	2596	1846	1310 ± 1500	−300	160	−2190
(i)	1120	775	−267	−384	52	167	0	300 ± 350	10	40	−280
Node:											
(a)	−3241	2545	−201	−1099	790	107	−525	−26 ± 6	0	−2	60
(b)	−3026	2274	−96	−1040	680	167	−512	−40 ± 9	1	−7	30
(c)	−2864	261	1168	−16	−480	−37	194	−52 ± 15	−4	−1	260
(d)	−4615	5068	−1992	−1173	2162	−1137	−355	131 ± 40	−33	52	470
(e)	−2240	−2037	−808	331	811	657	219	−1262 ± 25	4	12	260
(f)	194	145	82	42	20	9	4	65 ± 2	0	1	−35
(g)	−1716	300	511	−126	−207	60	96	−56 ± 5	4	6	90
(h)	−3334	−188	1667	489	−747	−441	278	−12 ± 30	20	4	560
(i)	−1181	67	288	3	−97	−11	36	−156 ± 40	−62	−69	35

Table 20.—*Equations of Condition for Odd Harmonics*

	J_3	J_5	J_7	J_9	J_{11}	J_{13}	(O–C)	I	II
Eccentricity:							10^{-6}		
(a)	−192.5	94.0	47.6	−76.3	21.9	25.2	4 ± 2	−2	0
(b)	−190.6	86.0	50.8	−71.5	16.1	26.6	2 ± 2	0	−1
(c)	−271.7	−165.1	138.6	89.6	−53.1	−43.7	8.6 ± 1.3	−0.1	0.1
(d)	−189.0	161.1	−10.2	−93.1	85.3	−16.1	−9 ± 14	−21	−19
(e)	−369.1	1036.4	1457.3	907.1	67.1	−481.1	−42 ± 6	0	−1
(f)	−293.0	−134.6	−51.0	−17.1	−4.9	−1.0	8 ± 4	3	−1
(g)	−214.6	−65.7	83.0	20.9	−30.2	−7.4	37 ± 20	34	33
(h)	−301.9	−312.1	136.1	220.5	−20.3	−125.9	22 ± 4	−4	−4
(i)	−202.0	−88.3	51.3	28.0	−12.2	9.5	18 ± 20	12	10
Inclination:							10^{-4}		
(a)	2890	−1420	−720	1140	−330	−380	−0.2 ± 0.8	0.1	0.4
(b)	3250	−1460	−870	1220	−280	−450	−0.4 ± 0.9	−0.1	0.1
(c)	160	100	−80	−50	30	30	−0.9 ± 0.8	−0.8	−0.8
(d)	1710	−1460	90	840	−770	150	−11 ± 7	−10	−10
(e)	70	−200	−290	−180	−10	90	−0.6 ± 1.3	−0.7	−0.7
(f)	−20	−10	0	0	0	0	1 ± 1	1	1
(g)	3190	980	−1230	−310	450	110	4 ± 3	4	5
(h)	100	110	−50	−70	10	40	−5 ± 3	−5	−5
(i)	3280	1430	−830	−450	200	150	−5 ± 9	−4	−4

Perigee:									
(a)	−64400	47200	11300	−42000	22510	13210	42 ± 8	−7	−5
(b)	−54900	41800	10900	−39100	19630	14600	50 ± 5	6	7
(d)	−123400	115950	−17320	−66120	73910	−22580	−14 ± 50	−141	−116
(g)	−50720	−7880	36420	3680	−21310	−1940	−20 ± 5	3	6
(i)	−41290	−13530	25230	10490	−10980	−6370	4 ± 100	6	8
Node:							10^{-4}		
(a)	−5300	−9700	13700	−1200	−9580	7330	−5 ± 5	−1	−2
(b)	−5900	−11150	14650	−300	−10800	7230	−7 ± 2	−1	−2
(c)	−220	−1490	−410	1000	480	−510	2 ± 4	0	0
(d)	−3550	−4240	11220	−7370	−3080	9310	14 ± 10	1	2
(e)	−80	−12390	−12630	−3900	4730	7500	4 ± 8	−9	−16
(f)	20	210	210	150	80	40	7 ± 7	7	7
(g)	−4520	−19460	−860	10990	1170	−5710	34 ± 6	16	16
(h)	−130	−1340	−760	820	980	−230	7 ± 5	4	4
(i)	−4440	−20940	−4580	7940	3450	−2720	88 ± 30	64	63

The two sets of solutions derived are the following:

Solution I (in units of 10^{-7})

$$\begin{array}{llll} dJ_2 = 1.65, & dJ_4 = 1.81, & dJ_6 = 2.56, & dJ_8 = -2.50, \\ \pm 6 & \pm 16 & \pm 30 & \pm 50 \\ & J_{10} = -0.54, & J_{12} = -3.57, & J_{14} = 1.79, \\ & \pm 50 & \pm 44 & \pm 63 \end{array} \tag{18}$$

Solution II (in units of 10^{-7})

$$\begin{array}{lll} dJ_2 = 1.50, & dJ_4 = 2.03, & dJ_6 = 2.03, \\ \pm 5 & \pm 18 & \pm 31 \\ dJ_8 = -1.29, & J_{10} = -1.55, & J_{12} = -2.94. \\ \pm 34 & \pm 45 & \pm 49 \end{array} \tag{19}$$

(b) Odd harmonics—As shown in table 20, we have 32 equations to determine 6 unknown coefficients of odd harmonics. At first $\Sigma(\mathrm{O-C})^2$ is $349(=32\times 3.3^2)$. This number comes down to $153(=28\times 2.3^2)$ after J_9 is solved, and to $42(=27\times 1.25^2)$ and to $39(=26\times 1.23^2)$ after J_{11} and J_{13}, respectively, are solved. Therefore, the inclusion of J_{13} does not reduce the residuals too much. Two sets of solutions are derived, one up to J_{11} and one up to J_{13}; that is,

Solution I (in units of 10^{-7})

$$\begin{array}{lll} dJ_3 = 0.31, & dJ_5 = -1.47, & dJ_7 = 1.36, \\ \pm 20 & \pm 25 & \pm 39 \\ J_9 = -1.67, & J_{11} = 3.02, & J_{13} = -1.14, \\ \pm 60 & \pm 35 & \pm 84 \end{array}$$

Solution II (in units of 10^{-7})

$$\begin{array}{lll} dJ_3 = 0.07, & dJ_5 = -1.22, & dJ_7 = 0.93, \\ \pm 11 & \pm 17 & \pm 22 \\ & J_9 = -0.75, & J_{11} = 2.96. \\ & \pm 17 & \pm 35 \end{array} \tag{20}$$

Table 20 gives the residuals based on solutions I and II for each datum. Residuals in the eccentricities of 1961 ν and 1962 $\beta\mu$, in the perigee of 1961 ν, and in the nodes of 1962 $\alpha\epsilon$ and 1962 $\beta\upsilon$ have much larger values than the standard errors. This may show that still higher-order harmonics are significant.

In this analysis parallactic terms are neglected in computing lunar perturbations. However, in the parallactic disturbing function there is a term,

$$\frac{45}{8}\sin i\cdot\sin\epsilon\left(1-\frac{5}{4}\sin^2 i\right)\left(1-\frac{5}{4}\sin^2\epsilon\right)ee'\left(1+\frac{3}{4}e^2\right)\sin\omega\cdot\sin\omega', \quad (21)$$

where ϵ is obliquity, e is lunar eccentricity, and ω' is lunar argument of perigee. Since ω' moves slowly, we must include this term if we treat observations of high-altitude satellites in the future.

5. RESULTS

The two sets of solutions derived in this paper are the following:

Solution I (units of 10^{-6})

$$\begin{array}{llll}
J_2 = & 1082.645, & J_3 = & -2.546, \\
 & \pm 6 & & \pm 20 \\
J_4 = & -1.649, & J_5 = & -0.210, \\
 & \pm 16 & & \pm 25 \\
J_6 = & 0.646, & J_7 = & -0.333, \\
 & \pm 30 & & \pm 39 \\
J_8 = & -0.270, & J_9 = & -0.353, \\
 & \pm 50 & & \pm 60 \\
J_{10} = & -0.054, & J_{11} = & 0.302, \\
 & \pm 50 & & \pm 35 \\
J_{12} = & -0.357, & J_{13} = & -0.114, \\
 & \pm 44 & & \pm 84 \\
J_{14} = & 0.179 & & \\
 & \pm 63 & &
\end{array} \quad (22)$$

Solution II

$$\begin{array}{llll}
J_2 = & 1082.630, & J_3 = & -2.559, \\
 & \pm 5 & & \pm 11 \\
J_4 = & -1.627, & J_5 = & -0.185, \\
 & \pm 18 & & \pm 17 \\
J_6 = & 0.593, & J_7 = & -0.376, \\
 & \pm 31 & & \pm 22
\end{array}$$

$$J_8 = -0.149, \quad J_9 = 0.039,$$
$$\pm 34 \qquad\qquad \pm 17$$

$$J_{10} = -0.155, \quad J_{11} = 0.296,$$
$$\pm 45 \qquad\qquad \pm 35$$

$$J_{12} = -0.294$$
$$\pm 49 \tag{23}$$

A. H. Cook (1964) recently derived values of J_2, J_4 and J_6 by using high satellites only, and his results show remarkable agreement with Solution I.

The flattening of the reference earth ellipsoid based on this value of J_2 is 1/298.252. The theoretical value of J_4 for the reference ellipsoid assumed to be in hydrostatic equilibrium is computed as -2.350×10^{-6}. The deviation of the geoid computed on the geopotential based on solution I is expressed as a function of geometric latitude:

$$\begin{aligned} h = &+0.8-18.3 \sin\beta-87.8 \sin^2\beta-119.1 \sin^3\beta+1042.5 \sin^4\beta \\ &+1191.7 \sin^5\beta-5074.2 \sin^6\beta-3636.7 \sin^7\beta+12{,}668.0 \sin^8\beta \\ &+5230.8 \sin^9\beta-16{,}676.3 \sin^{10}\beta-3556.4 \sin^{11}\beta+10{,}913.0 \sin^{12}\beta \\ &+926.8 \sin^{13}\beta-2791.3 \sin^{14}\beta \text{ (in meters).} \end{aligned} \tag{24}$$

Figure 1 shows the value of h as a function of β based on this equation. The value of geoid height h in the north pole is 13.5 meters, which is the maximum value, and is -24.1 meters in the south pole.

In the solutions (22) and (23), the values of J_n do not tend to converge to zero as n increases. However, if n is large enough, J_n should take a very small value. Otherwise the gravity expression, which is derived by differentiating the potential with respect to the radius, may give a very great difference of gravities between the equator and the poles and between the north and south poles.

To determine how strong or weak the solutions (22) and (23) are, the correlation coefficients in my determinations are shown in tables 21 and 22. The tables indicate that these solutions are derived from rather strongly correlated equations of condition. Therefore, in the future we must use both low and high satellites having the same inclination.

However, to determine the orbital elements of low satellites with high inclinations we need observations from high latitudes. As I mentioned earlier, I could not assign a large weight to the node equation of 1961*o* to determine even-order coefficients, because the inclination could not be determined with sufficient accuracy. Also, I must mention that I did not use satellites with inclinations below 28°, between 50° and 67°, or between 67° and 85° in this determination.

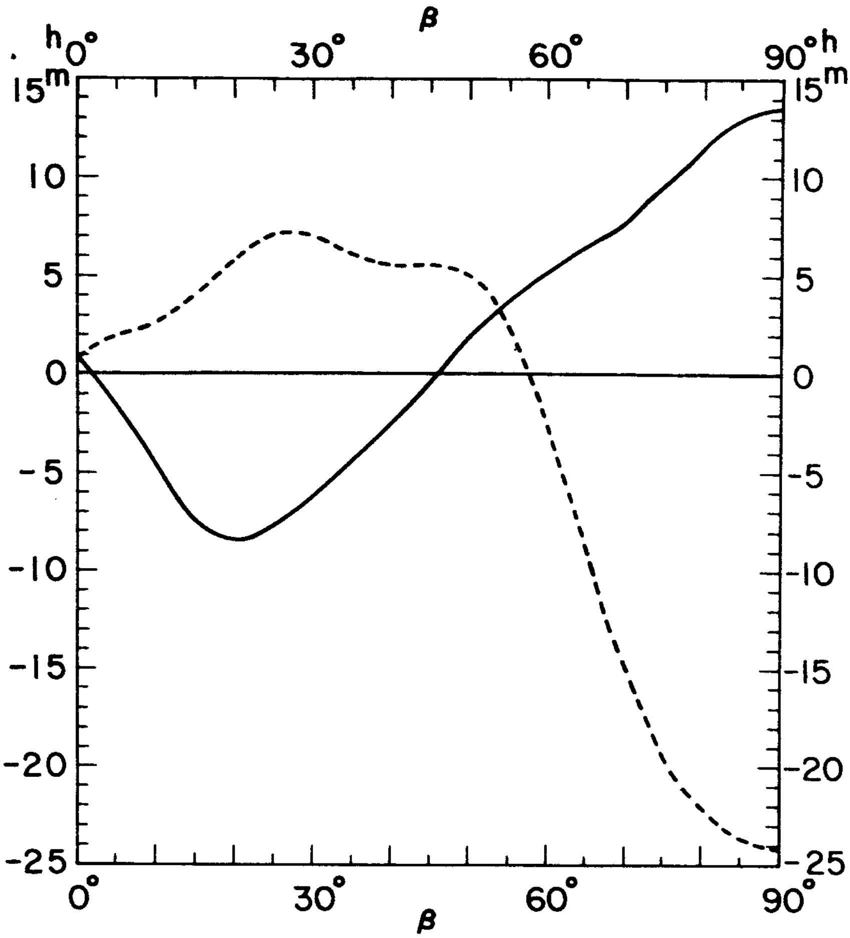

Figure 1.—Geoid height (*h*) as a function of geometric latitude (*β*). Solid line shows geoid height in Northern Hemisphere and broken line shows that in the Southern Hemisphere.

However, I believe that the present determination is much more reliable than the previous one, since the data themselves are more reliable, both because of the number of observations and because I included some satellites that were not used in the previous determination.

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Table 21.—*Correlation Coefficients for Even Orders*

	J_2	J_4	J_6	J_8	J_{10}	J_{12}	J_{14}		J_2	J_4	J_6	J_8	J_{10}	J_{12}
J_2	1.00	−0.60	0.80	−0.89	0.79	−0.71	0.83	J_2	1.00	−0.40	0.63	−0.60	0.49	−0.57
J_4		1.00	−0.86	0.80	−0.85	0.91	−0.47	J_4		1.00	−0.82	0.84	−0.84	0.88
J_6			1.00	−0.79	0.96	−0.88	0.60	J_6			1.00	−0.65	0.94	−0.83
J_8				1.00	−0.80	0.84	−0.84	J_8				1.00	−0.54	0.90
J_{10}					1.00	−0.80	0.70	J_{10}					1.00	−0.73
J_{12}						1.00	−0.50	J_{12}						1.00
J_{14}							1.00							

Table 22.—*Correlation Coefficients for Odd Orders*

	J_3	J_5	J_7	J_9	J_{11}	J_{13}		J_3	J_5	J_7	J_9	J_{11}
J_3	1.00	−0.93	0.98	−0.94	0.48	−0.86	J_3	1.00	−0.85	0.93	−0.85	0.75
J_5		1.00	−0.96	0.86	−0.69	0.75	J_5		1.00	−0.92	0.81	−0.92
J_7			1.00	−0.92	0.57	−0.82	J_7			1.00	−0.92	0.84
J_9				1.00	−0.27	0.97	J_9				1.00	−0.68
J_{11}					1.00	−0.12	J_{11}					1.00
J_{13}						1.00						

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